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# Color-character of uncolorable cubic graphs

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ABSTRACT

Let $G = (V, E)$ be a cubic graph with chromatic index 4 and $c : E \longrightarrow \{0, 1, 2, 3\}$ a proper 4-edge-coloring of $G$. Let $E_i = \{e \in E \mid c(e) = i\}$ and $\circ(c) = \min\{|E_i| \mid i = 0, 1, 2, 3\}$. If $\mathcal{C}(G)$ denotes all the proper 4-edge-colorings of $G$, then $m(G) = \min_{c\in\mathcal{C}(G)}\{\circ(c)\}$ is defined to be the *color-character* of $G$. In this work, we prove that $m(G)$ is a constant under some operations, and give a relation between $m(G)$ and another parameter of $G$.


## 1. Introduction

In this work, by a *graph* we mean a finite graph with multiple edges and without loops. The *chromatic index* of a graph $G$, denoted by $\chi'(G)$, is the smallest integer $k$ for which a proper $k$-edge-coloring exists and a *$k$-coloring* is defined to be a proper $k$-edge-coloring. It is well known that the chromatic index of a cubic graph is either 3 or 4. Graphs with chromatic index 4 are called *uncolorable* graphs while the others are called *colorable*. Uncolorable graphs are of great interest, since there are counterexamples to some hard conjectures, e.g. Tutte's 5-flow Conjecture, the Cycle Double-Cover Conjecture (CDCC) (see e.g. [1,2]); then they must be uncolorable. These famous conjectures are obstructed by uncolorable cubic graphs. Moreover, the Four-Color Theorem is equivalent to the statement that every bridgeless cubic planar graph is colorable (see e.g. [3,4]). Motivated by these researches we focus our attention on the study of uncolorable cubic graphs.

We consider a parameter: the *color-character* of uncolorable cubic graphs. Let $G$ be an uncolorable cubic graph. Suppose $c$ is a 4-coloring of $G$ and $E_i = \{e \in E \mid c(e) = i\}$, $i = 0, 1, 2, 3$. Let $\circ(c) = \min\{|E_i| \mid i = 0, 1, 2, 3\}$ and define $m(G) = \min_{c\in\mathcal{C}(G)}\{\circ(c)\}$ to be the *color-character* of $G$, where $\mathcal{C}(G)$ consists of all the 4-colorings of $G$. If $G$ is a cubic graph with $\chi'(G) = 3$, we define $m(G) = 0$. $c \in \mathcal{C}(G)$ is said to be a *character-coloring* of $G$ if $\circ(c) = m(G)$. Intuitively, $m(G)$ is the minimal number of edges which have to be removed from $G$ to obtain a graph with chromatic index 3.

Steffen proved some properties of the color-character of uncolorable cubic graphs in [5] (in [5], $m(G)$ is called *color number* and denoted by $c(G)$), and discussed the relationship between the color-character and oddness of uncolorable cubic graphs in [6].

In this work, we prove that $m(G)$ is an invariant under some operations of $G$ and give a relationship between $m(G)$ and another parameter of $G$. Note that color-character provides a way to classify uncolorable cubic graphs. So it enables us to study those graphs class by class, and especially to verify those famous conjectures.

## 2. Notation and known results

Let $G$ be an uncolorable cubic graph. Suppose $c$ is a character-coloring of $G$ using colors 0, 1, 2, 3. Without loss of generality, in this work we always assume that $m(G) = \circ(c) = |E_0|$. For $v \in V(G)$, let $E(v) = \{e \in E(G) \mid v \text{ is an end of } e\}$ and

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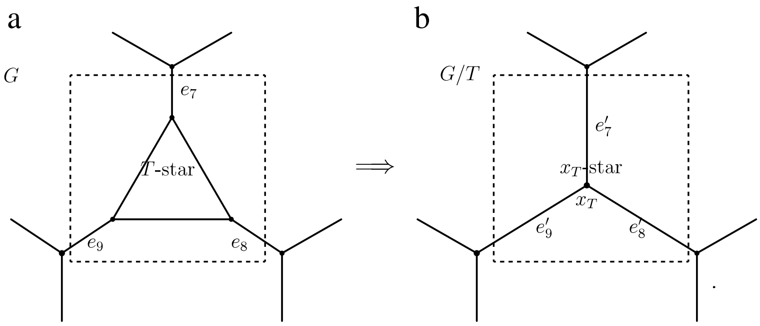


**Fig. 1.** Getting $G/T$ from $G$ by $\Delta$-reduction.

$c(v) = \{c(e) \mid e \in E(v)\}$. For any $e = xy \in E_0$, define $t(e) = (c(x) \cap c(y)) \setminus \{0\}$ and $\overline{t(e)} = c(x) \bigtriangledown c(y)$, where "$\bigtriangledown$" is the symmetric difference. Since $c$ is a character-coloring, it is easy to see that $|\overline{t(e)}| = 2$. Define $c(E) = \{c(e) \mid e \in E\}$.

We modify the definition of an edge to create a *semiedge* which is like an edge but has just one end vertex; pairs of semiedges will combined to form edges. A *multipole* $M = (V, E, S)$ consists of a set of vertices $V = V(M)$, a set of edges $E = E(M)$ and a set of semiedges $S = S(M)$. If $|S(M)| = k$, then $M$ is also called a *k-pole*. An *n-coloring* of $M = (V, E, S)$ is a mapping $c$ from $E \cup S$ into $\{1, 2, \dots, n\}$ such that $c(e_1) \neq c(e_2)$ for all adjacent (semi) edges $e_1$, $e_2$. Let $f_1$ and $f_2$ be two semiedges of $M$, and incident with $v_1$ and $v_2$ respectively. We say that $M'$ is obtained from $M$ by *identifying* semiedges $f_1$ and $f_2$ if these two semiedges are replaced by an edge $v_1v_2$.

For a colored graph $G$ and colors $x$, $y$ we define an *$(x, y)$-path* to be a path in $G$ whose edges are colored with $x$ and $y$ alternately.

A triangle $T$ in cubic graph $G$ along with the pendant edges $e_7$, $e_8$ and $e_9$ is called the *$T$-star*, which is denoted by $S(T)$ (see Fig. 1(a)). Contracting $T$ into a new vertex $x_T$, we get a new cubic graph $G/T$ from $G$ (see Fig. 1(b)), where the $T$-star $S(T)$ becomes the $x_T$-star at vertex $x_T$ which is incident with three new edges $e'_7$, $e'_8$ and $e'_9$ and the corresponding operation is called a $\triangle$-reduction on $T$.

Suppose $P = v_0 \dots v_k$ is a path in $G$, where the degree of $v_0$ and $v_k$ is 3 and the degree of $v_1, \dots, v_{k-1}$ is 2. We delete $v_1, \dots, v_{k-1}$ and add a new edge $v_0v_k$ to $G$. Such an operation is called *suppressing* $v_1, \dots, v_{k-1}$. Let $G$ be a cubic graph and suppose $Q = v_1e_1v_2e_2v_3e_3v_4e_4v_1$ is a square (4-cycle) of $G$. We construct a new cubic graph by deleting the edges $e_1$ and $e_3$ and suppressing the two resulting paths of two vertices of degree 2. Such a graph is denoted by $G \oplus_{\square} \{e_1, e_3\}$, and we call the corresponding operation a $\square$-reduction. We prove that either $G \oplus_{\square} \{e_1, e_3\}$ or $G \oplus_{\square} \{e_2, e_4\}$ has the same color-character as $G$.

Let $\mathbb{G}_n$ be the collection of cubic graphs of order $n$. Then $\mathbb{G}_n$ can be partitioned into the cubic graphs with chromatic index 3, denoted by $\mathbb{G}_n^{(1)}$, and the cubic graphs with chromatic index 4, denoted by $\mathbb{G}_n^{(2)}$.

Suppose $G$ is a cubic graph in $\mathbb{G}_n$, and $e_1$ and $e_2$ are two edges of $G$. Subdividing $e_i$ in $G$ by a new vertex $u_i$ $(i = 1, 2)$ and adding a new edge $u_1u_2$, we get a new cubic graph $G(e_1, e_2)$ from $G$ (note that $e_1 = e_2$ is permitted; in this case $u_1$ and $u_2$ will be two subdividing vertices on the same edge $e_1$). Since this process is reversible, $\mathbb{G}_{n+2} = \{G(e_1, e_2) \mid e_1, e_2 \in G \in \mathbb{G}_n\}$. It is convenient to regard $G(e_1, e_2)$ as an operation on $G$ and $\{e_1, e_2\}$, which is denoted by $G(e_1, e_2) = G \mid - \mid \{e_1, e_2\}$.

Let $S$ be a set of some pairs of edges of $G$, and $E_S = \bigcup_{\{e_i, e_j\} \in S} \{e_i, e_j\}$. For each edge $e \in E_S$, let $n_e$ be the number of pairs of $S$ containing $e$. Suppose $S$ satisfies the following two conditions:

(i) $e_i \neq e_j$, for all $\{e_i, e_j\} \in S$;
(ii) $n_e \leqslant 2$ for all $e \in E_S$, and there is at most one edge, say $e_0$, in $E_S$ such that $n_{e_0} = 2$.

Then we construct $G(S)$ as follows. Subdivide $e \in E_S \setminus \{e_0\}$ by a new vertex $u_e$, and subdivide $e_0$ (if it exists) by two new vertices $u_{e_0}$ and $u'_{e_0}$. For each pair $\{e_i, e_j\} \in S$, $e_i, e_j \neq e_0$, add a new edge $u_{e_i}u_{e_j}$ to $G$, and for the two pairs $\{e_i, e_0\}$ and $\{e_0, e_j\}$, add two new edges $u_{e_i}u_{e_0}$ and $u'_{e_0}u_{e_j}$, or $u_{e_i}u'_{e_0}$ and $u_{e_0}u_{e_j}$ to $G$. Note that if such $e_0$ exists, then $G(S)$ are two graphs. Clearly, $G(S) \in \mathbb{G}_{n+2k}$ if $G \in \mathbb{G}_n$ and $|S| = k$. For $G \in G_n$, let $S$ be a minimum set satisfying (i), (ii) and $G(S) \in \mathbb{G}_{n+2k}^{(1)}$, for some $k$. We define $s(G) = |S|$ (if $\chi'(G) = 3$, we define $s(G) = 0$). In Section 4, we prove that $s(G) = \lceil \frac{m(G)}{2} \rceil$. Namely, for any $G \in \mathbb{G}_n^{(2)}$, there exists a set of $\lceil \frac{m(G)}{2} \rceil$ pairs of edges of $G$, say $S$, that satisfies (i), (ii) and $G(S) \in \mathbb{G}^{(1)}_{n+2\lceil \frac{m(G)}{2} \rceil}$. And $\chi'(G(S))$ will never be 3 if $|S| < \lceil \frac{m(G)}{2} \rceil$.

Here are some known results about cubic graphs. These results are famous or will be used in this work. We start with the Parity Lemma of [7], which is a very useful tool for proofs in the field of uncolorable cubic graphs.

**Lemma 2.1** *([7]). Let $G$ be a colorable cubic graph that has been 3-colored with colors* 1, 2 *and* 3. *If a cutset consisting of $n$ edges contains $n_i$ edges of color $i$, for $i = 1, 2, 3$, then*

$$n_1 \equiv n_2 \equiv n_3 \equiv n \pmod 2.$$

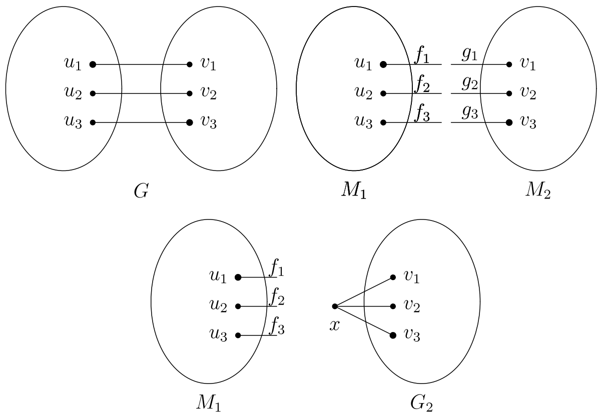


**Fig. 2.** Graphs of Theorem 3.1.

We define for $i = 1, 2, 3$ the subsets $H_i$ of $E_0$ as follows: $H_i = \{e \mid t(e) = i\}$. It is easy to see that $H_i \cap H_j = \emptyset$ if $i \neq j$, and $E_0 = H_1 \cup H_2 \cup H_3$. Lemma 2.2 in [5] is obtained from Lemma 2.1.

**Lemma 2.2** ([5]). *Let $G$ be an uncolorable cubic graph, and $c$ be a character-coloring of $G$ with $E_0 = H_1 \cup H_2 \cup H_3$. Then* $|H_1| \equiv |H_2| \equiv |H_3| \equiv m(G) \pmod 2$.

And in [5], Steffen also proved the following two lemmas, which will be used in this work.

**Lemma 2.3** ([5]). *Let $e = vw \in E_0$ and $\overline{t(e)} = \{y, z\}$. Then there is a $(y, z)$-path from $v$ to $w$.*

Let $e = vw$ be the edge of Lemma 2.3; then $e$ together with the $(y, z)$-path from $v$ to $w$ forms a cycle $C_e$ of odd length in $G$. We will say that $e$ *admits* cycle $C_e$.

**Lemma 2.4** ([5]). *Let $e_1$ and $e_2$ be two different edges of $E_0$. Then $c_{e_1}$ and $c_{e_2}$ are disjoint.*

Lemma 2.5 in [8] shows that the chromatic index of a cubic graph is a constant under $\triangle$-reduction, while Lemma 2.6 in [3] is a result about $\square$-reduction.

**Lemma 2.5** ([8]). *Let $G$ be a cubic graph. If $G'$ is obtained from $G$ via a sequence of $\triangle$-reductions, then $\chi'(G') = \chi'(G)$.*

**Lemma 2.6** ([3]). *If $G$ is a cubic graph and $Q = v_1e_1v_2e_2v_3e_3v_4e_4v_1$ is a square of $G$, then $G$ is 3-colorable if and only if one of $G \oplus_{\square} \{e_1, e_3\}$ and $G \oplus_{\square} \{e_2, e_4\}$ is.*

In Section 3, we consider the color-character of cubic graphs under $\triangle$-reductions and $\square$-reductions. Then in Section 4, we give a relation between $s(G)$ and $m(G)$.

## 3. $m(G)$ is an invariant under some operations

**Theorem 3.1.** *Let $G$ be an uncolorable cubic graph, and let $\{u_1v_1, u_2v_2, u_3v_3\}$ be a 3-edge-cut of $G$. For $i = 1, 2, 3$, replace $u_iv_i$ by two semiedges $f_i$ and $g_i$ which are incident to $u_i$ and $v_i$ respectively, and we get two 3-poles $M_1$ and $M_2$. See Fig. 2. If $\chi'(M_1) = 3$, and $G_2$ is the cubic graph obtained from $M_2$ by joining the three semiedges of $M_2$ to a new vertex $x$, then $m(G) = m(G_2)$.*

**Proof.** Without loss of generality, assume that $\{u_1, u_2, u_3\} \subseteq V(M_1)$ (Fig. 2). Let $c_2$ be a character-coloring of $G_2$ and $m(G_2) = \circ(c_2) = |E_0|$. If $0 \in c_2(x)$, say $c_2(xv_1) = 0$, since $c_2$ is a character-coloring, there exists a color $i \in c_2(v_1)$ and $i \notin c_2(x)$, say $c_2(v_1y) = i$. Then let $P$ be the longest $(0, i)$-path that contains $xv_1$ and $v_1y$ in $M_2$. Interchanging colors 0 and 1 along $P$ we get a character-coloring of $G_2$ such that no edge incident with $x$ is colored with 0. So we may assume that there is no edge incident with $x$ that is colored with 0 under $c_2$. Lemma 2.1 implies that for any 3-coloring of $M_1$, $f_1$, $f_2$ and $f_3$ must be colored with three distinct colors. Moreover, there is a 3-coloring $c_1$ of $M_1$ where $c_1(f_i) = c_2(v_ix)$, $i = 1, 2, 3$. So we can get a 4-coloring $c$ of $G$ from $c_1$ and $c_2$ such that $\circ(c) = \circ(c_2)$. So $m(G) \leqslant m(G_2)$.

Suppose $c$ is a character-coloring of $G$ and $m(G) = \circ(c) = |E_0|$. Let $A = \{c(u_1v_1), c(u_2v_2), c(u_3v_3)\}$. Let $c_1$ be a coloring of $M_1$ derived from $c$, where $c_1(e) = c(e)$ for $e \in E(M_1)$ and $c_1(f_i) = c(u_iv_i)$. If no edge and no semiedge of $M_1$ are colored with 0 under $c_1$, then by Lemma 2.1, $A = \{c_1(f_1), c_1(f_2), c_1(f_3)\} = \{1, 2, 3\}$. Then we can naturally derive a 4-coloring $c_2$ of $G_2$ from $c$, where $c_2(e) = c(e)$ for $e \in E(M_2)$, and $c_2(v_ix) = c(u_iv_i)$. So $\circ(c_2) \leqslant \circ(c)$, and thus $m(G_2) \leqslant m(G)$. Now suppose that there is at least one edge or one semiedge of $M_1$ which is colored with 0 under $c$. Let $c_{v_i} = \{c(e) \mid e \text{ is an edge incident with } v_i \text{ in } M_2\}$.

**Case 1:** $0 \notin c_{v_i}$ for $i = 1, 2, 3$.

Suppose that $c_{v_1}$, $c_{v_2}$ and $c_{v_3}$ are distinct sets. Then we can derive a 4-coloring $c_2$ of $G_2$ from $c$, where $c_2(e) = c(e)$ for $e \in E(M_2)$, and $c_2(v_i x) = \{1, 2, 3\} \setminus c_{v_i}$. So $\circ(c_2) \leqslant \circ(c)$, and thus $m(G_2) \leqslant m(G)$. If exactly two of them are alike, then there is a 4-coloring $c_2$ of $G_2$ where $c_2(e) = c(e)$ for $e \in E(M_2)$ and only one edge incident with $x$ is colored with 0. Since there is at least one edge or one semiedge of $M_1$ which is colored with 0 under $c$, $\circ(c_2) \leqslant \circ(c)$, and thus $m(G_2) \leqslant m(G)$. If $c_{v_1} = c_{v_2} = c_{v_3}$, say $c_{v_1} = c_{v_2} = c_{v_3} = \{1, 2\}$, then in $M_2$, there exists a $(1, 3)$-path starting from one vertex of $v_1$, $v_2$ and $v_3$, but that does not end in $v_1$, $v_2$ and $v_3$. Interchanging colors 1 and 3 along this path yields the above case. So $m(G_2) \leqslant m(G)$.

**Case 2:** $0 \in c_{v_i}$ for some $i = 1, 2$ or 3.

Without loss of generality, we assume that $0 \in c_{v_3}$. If there exists a set, say $c_{v_1}$, such that $0 \notin c_{v_1}$, then we can define a 4-coloring $c_2$ of $G_2$ where $c_2(e) = c(e)$ for $e \in E(M_2)$, $c_2(v_1 x) = 0$, $c_2(v_2 x) \in \{1, 2, 3\} \setminus c_{v_2}$ and $c_2(v_3 x) \in \{0, 1, 2, 3\} \setminus (c_{v_3} \cup c_2(v_2 x))$. Then $\circ(c_2) \leqslant \circ(c)$, and thus $m(G_2) \leqslant m(G)$. Suppose $0 \in c_{v_1}$, $c_{v_2}$ and $c_{v_3}$. Let $y, z$ be the two neighbors of $v_1$ in $M_2$, and assume $c(v_1 y) = 0$. Since $c$ is a character-coloring of $G$, there is an edge, say $yu$ ($u \neq v_1$), incident with $y$ such that $c(yu) \neq c(v_1 z)$ and $c(v_1 u_1)$. Suppose that $c(yu) = 1$. Then let $P$ be the longest $(0, 1)$-path that contains $v_1 y$ and $yu$ in $M_2$. Interchanging colors 0 and 1 along this path yields the above case. So $m(G_2) \leqslant m(G)$. □

**Corollary 3.1.** *If $G$ is a cubic graph, $T$ is a triangle in $G$ and $G/T$ is defined above, then*

$$m(G) = m(G/T).$$

**Proof.** If $m(G) = 0$, the corollary holds by Lemma 2.5. If $m(G) > 0$, let $M_1$ be the $T$-star of $G$. Then the proposition follows from Theorem 3.1 directly. □

From Corollary 3.1, we immediately have:

**Corollary 3.2.** *Let $G$ be a cubic graph. If $G'$ is obtained from $G$ via a sequence of $\triangle$-reductions, then $m(G') = m(G)$.*

Corollary 3.2 implies that the color-character of a cubic graph is a constant under the $\triangle$-reduction. But it is not always true for $\square$-reduction since a square inserted along any two non-adjacent edges of the Petersen graph yields a graph with chromatic index 3. But we have the following weaker theorem.

**Theorem 3.2.** *If $G$ is a cubic graph and $Q = v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_1$ is a square of $G$, then either $m(G \oplus_\square \{e_1, e_3\}) = m(G)$ or $m(G \oplus_\square \{e_2, e_4\}) = m(G)$.*

**Proof.** If $m(G) = 0$, the theorem holds by Lemma 2.6. So assume that $\chi'(G) = 4$. Let $G \oplus_\square \{e_1, e_3\} = G_1$ and $G \oplus_\square \{e_2, e_4\} = G_2$. Note that $v_i$ has only one neighbor which is not in $V(Q)$; denote this neighbor by $u_i$.

We can reconstruct $G$ from $G_1$, and it is easy to see that any proper coloring $c_1$ of $G_1$ can be extended to a proper coloring $c$ of $G$ such that $\circ(c) = \circ(c_1)$. So $m(G) \leqslant m(G_1)$. Similarly, $m(G) \leqslant m(G_2)$. Now, suppose $c$ is a character-coloring of $G$. In the following, we prove that there is a 4-coloring $c'$ of $G_1$ or $G_2$ such that $\circ(c') \leqslant \circ(c)$, and thus $m(G_1) \leqslant m(G)$ or $m(G_2) \leqslant m(G)$. Let $e'_i = u_i v_i$, $i = 1, 2, 3, 4$, and $E' = \{e'_1, e'_2, e'_3, e'_4\}$. By Lemma 2.4 we can always assume that $0 \notin c(E')$. We consider the following three cases.

**Case 1**: $c(e'_1) = c(e'_2) = c(e'_3) = c(e'_4)$.

In this case, the result is trivial.

Let $u'_i$ and $u''_j$ be the two neighbors of $u_i$ other than $v_i$, $E^i = \{u_i u'_i, u_i u''_i\}$, and $D = \bigcup_{i=1}^{4} E^i$.

**Case 2**: Exactly three edges of $E'$ have the same color.

Assume, without loss of generality, that $c(e'_1) = 1$ and $c(e'_2) = c(e'_3) = c(e'_4) = 2$. Clearly, in this case, $0 \in c(E(Q))$. We claim that $0 \notin c(D)$. In fact, if $0 \in E^1$ and $c(u_1 u'_1) = 0$, we can find a character-coloring $c_0$ of $G$ corresponding to $c$, where $c_0(v_1 v_2) = 0$, $c_0(v_2 v_3) = c_0(v_4 v_1) = 3$, $c_0(v_3 v_4) = 1$, and $c_0(e) = c(e)$ for other edges of $G$. Under character-coloring $c_0$, cycle $c_{v_1 v_2}$ contains vertex $u_1$, which must be contained in cycle $c_{u_1 u'_1}$. This is a contradiction to Lemma 2.4. So $0 \notin c(E^1)$. Similarly, $0 \notin c(E^2)$. And since we can also find a character-coloring of $G$ under which $v_1 v_4$ is colored with 0 and $v_1 v_2$, $v_3 v_4$ are both colored with 3, $0 \notin c(E^3)$ and $0 \notin c(E^4)$. So $0 \notin c(D)$. Then we can get a 4-coloring $c'$ of $G_1$ where $c'(u_1 u_2) = 0$, $c'(u_3 u_4) = 2$, and $c'(e) = c(e)$ for each $e \in E(G) \cap E(G_1)$; since $0 \in c(E(Q))$, $\circ(c') \leqslant \circ(c)$. (Similarly, such a $c'$ exists for $G_2$.)

**Case 3**: No three edges of $E'$ have the same color.

**Subcase 3.1**. Exactly one pair of edges of $E'$ have the same color.

First, suppose that $c(e'_i) = c(e'_{i+1})$ for some $i \in \{1, 2, 3, 4\}$ (when $i = 4$, let $e'_{i+1} = e'_1$). Without loss of generality, assume that $c(e'_1) = c(e'_2) = 1$, $c(e'_3) = 2$ and $c(e'_4) = 3$. It is easy to see that $0 \in c(E(Q))$. We claim that $0 \notin c(E^3)$, $c(E^4)$. Otherwise, suppose $0 \in c(E^3)$ and $c(u_3 u'_3) = 0$. Then we can find a character-coloring $c_0$ of $G$ corresponding to $c$, where $c_0(v_1 v_2) = 3$, $c_0(v_2 v_3) = 0$, $c_0(v_3 v_4) = 1$, $c_0(v_4 v_1) = 2$, and $c_0(e) = c(e)$ for other edges of $G$. Under character-coloring $c_0$, cycle $c_{v_2 v_3}$ contains vertex $u_3$, which must be contained in cycle $c_{u_3 u'_3}$. This is a contradiction to Lemma 2.4. So $0 \notin c(E^3)$. Similarly, we can find a character-coloring $c'_0$ of $G$ such that $c'_0(v_1 v_2) = 2$, $c'_0(v_2 v_3) = 3$, $c'_0(v_3 v_4) = 1$, $c'_0(v_4 v_1) = 0$, and $c'_0(e) = c(e)$ for other edges of $G$. So $0 \notin c(E^4)$. Then we can derive a 4-coloring $c'$ of $G_1$, where $c'(u_1 u_2) = 1$, $c'(u_3 u_4) = 0$, and $c'(e) = c(e)$ for each $e \in E(G) \cap E(G_1)$. So $\circ(c') \leqslant \circ(c)$. Thus $m(G_1) \leqslant m(G)$.

Now suppose that $c(e'_i) = c(e'_{i+2})$, $i = 1$ or 2; assume $c(e'_1) = c(e'_3) = 1$, $c(e'_2) = 2$ and $c(e'_4) = 3$. Then it is easy to see that $c_0(v_1v_2) = 0$, $c_0(v_2v_3) = 3$, $c_0(v_3v_4) = 0$, $c_0(v_4v_1) = 2$, and $c_0(e) = c(e)$ for other edges in $G$ is a character-coloring of $G$. By Lemma 2.4, this case will not happen.

**Subcase 3.2**. There are two pairs of edges of $E'$ such that the members of each pair have the same color.

If the two pairs are $(e'_1, e'_2)$ and $(e'_3, e'_4)$ or $(e'_1, e'_4)$ and $(e'_2, e'_3)$, then the result holds trivially. So assume that the two pairs are $(e'_1, e'_3)$ and $(e'_2, e'_4)$. Suppose that $c(e'_1) = c(e'_3) = 1$ and $c(e'_2) = c(e'_4) = 2$. It is easy to see that $c_0(v_1v_2) = c_0(v_3v_4) = 0$, $c_0(v_2v_3) = c_0(v_4v_1) = 3$, and $c_0(e) = c(e)$ for other edges in $G$ is a character-coloring of $G$. As above, we can observe that $4 \notin c(D)$. So let $c'$ be a coloring of $G$ such that $c'(u_1u_2) = c'(u_3u_4) = 0$, and $c'(e) = c(e)$ for each $e \in E(G) \cap E(G_1)$. Since there are two edges of $E(Q)$ colored 0 under $c$, $\circ(c') \leqslant \circ(c)$. (Similarly, such a $c'$ exists for $G_2$.) This completes the proof of the theorem. □

## 4. Relation between $s(G)$ and $m(G)$

Let $G$ be an uncolorable cubic graph. We can see from the definition of $m(G)$ that it is hard to ascertain its color-character. So we wish to establish relations between $m(G)$ and other parameters of $G$. In this section, we prove a relation between $s(G)$ and $m(G)$ which is given by Corollary 4.1.

As we introduced in Section 2, let $S$ be a set of some pairs of edges of $G$, and $E_S = \bigcup_{\{e_i,e_j\}\in S}\{e_i, e_j\}$. For each edge $e \in E_S$, let $n_e$ be the number of pairs of $S$ containing $e$. Then we state the following two conditions:

(i) $e_i \neq e_j$, for all $\{e_i, e_j\} \in S$;
(ii) $n_e \leqslant 2$ for all $e \in E_S$, and there is at most one edge, say $e_0$, in $E_S$ such that $n_{e_0} = 2$.

**Theorem 4.1.** *Let $G \in \mathbb{G}_n^{(2)}$ and $S$ be a set satisfying conditions* (i) *and* (ii). *If $\chi'(G(S)) = 3$ and $|S| = k$, then $m(G) \leqslant 2k$.*

**Proof.** First, we construct a new cubic graph $G'$ from $G(S)$. Let $e_1 = xy$ and $e_2 = xz$ be two adjacent edges in $E_S = \bigcup_{\{e_i,e_j\}\in S}\{e_i, e_j\}$. Let $u_1,\ u_2 \in V(G(S))$ be two vertices subdividing $e_1$ and $e_2$, and $xu_1,\ xu_2 \in E(G(S))$. Again subdivide $xu_1$ and $xu_2$ by $v_1$ and $v_2$ respectively, and add a new edge $v_1v_2$ to $G(S)$. Treat any two adjacent edges in $E_S$ like this, with the resulting graph denoted by $G'$. Then $\chi'(G') = \chi'(G(S)) = 3$ by Lemma 2.5. Now, let $E' = \{e = uv \mid u, v \in V(G(S)) \setminus V(G)\}$ and $G_0 = G' - E'$. Let $\mathscr{P} = \{P \mid P$ is a path of $G_0$, the degree of its two ends is 3 and the degree of its internal vertices is exactly 2$\}$. By replacing each $P \in \mathscr{P}$ with an edge we get a new cubic graph $G''$. Then $G$ is a graph obtained from $G''$ via a sequence of $\triangle$-reductions. Denote the edge set corresponding to $\mathscr{P}$ by $E_P$. Then $E_P$ is an independent edge set of $G''$ and $|E_P| = |E_S| \leqslant 2k$. Let $c'$ be a proper 3-edge-coloring of $G'$ using color 1, 2 and 3. Then we can obtain a proper 4-edge-coloring $c = \{E_0, E_1, E_2, E_3\}$ of $G''$, which derives from $c'$ as follows:

$$c(e) = \begin{cases} c'(e) & \text{if } e \notin E_P; \\ 0 & \text{if } e \in E_P. \end{cases}$$

Then $|E_0| = |E_P| \leqslant 2k$, so $m(G'') \leqslant 2k$. By Corollary 3.2, $m(G) = m(G'')$. Thus $m(G) \leqslant 2k$. □

**Theorem 4.2.** *If $G \in \mathbb{G}_n^{(2)}$, then there exists a set $S$ satisfying conditions* (i), (ii) *and $|S| = \lceil \frac{m(G)}{2} \rceil$ such that $\chi'(G(S)) = 3$.*

**Proof.** Suppose $c$ is a character-coloring of $G$ and $m(G) = \circ(c) = |E_0|$. If $H_i$ $(i = 1, 2, 3)$ is as defined in Section 2, then $|H_1| \equiv |H_2| \equiv |H_3| \equiv m(G) \pmod 2$ by Lemma 2.2. If $m(G)$ is even, then $|H_1|$, $|H_2|$ and $|H_3|$ are all even. Let $m(G) = 2n$ and $E_0 = \{e_1, e_2, \ldots, e_{2n}\}$ where $e_i = v_iw_i$ for $i = 1, 2, \ldots, 2n$. For each $i$, subdivide $e_i$ by a new vertex $u_i$ and add a semiedges $f_i$ on $u_i$. If $t(e_i) = x$ and $\overline{t(e_i)} = \{y, z\}$, color $u_iv_i$ and $u_iw_i$ in an appropriate way with colors $y$ and $z$ and $f_i$ with colors $x$. Then we get a 3-colored $2n$-pole $M$, and denote this coloring of $M$ by $c'$. Since each of $|H_1|$, $|H_2|$ and $|H_3|$ is even, we can partition $f_1, \ldots, f_{2n}$ into $n$ pairs in an appropriate way such that for each pair $(f_i, f_j)$, $c'(f_i) = c'(f_j)$. Then by identifying these pairs, we get a 3-colorable cubic graph. And $S = \{(e_i, e_j) \mid f_i$ and $f_j$ are in some pair$\}$ is the set desired.

If $m(G)$ is odd, then $H_1$, $H_2$ and $H_3$ are all odd. Let $m(G) = 2n + 1$ and $E_0 = \{e_1, e_2, \ldots, e_{2n+1}\}$ where $e_i = v_iw_i$ for $i = 1, 2, \ldots, 2n + 1$. Assume, without loss of generality, that $t(e_1) = 1$ $t(e_2) = 2$ and $t(e_3) = 3$. For each $i \geqslant 2$, subdivide $e_i$ by a new vertex $u_i$ and add a semiedge $f_i$ on $u_i$. Subdivide $e_1$ by two new vertices $u_1$ and $u'_1$, and add two semiedges $f_1, f'_1$ on $u_1, u'_1$ respectively. Then we get a $(2n+2)$-pole $M$. For each $i \geqslant 2$, if $t(e_i) = x$ and $\overline{t(e_i)} = \{y, z\}$, color $u_iv_i$ and $u_iw_i$ in an appropriate way with colors $y$ and $z$ and $f_i$ with colors $x$. And for $i = 1$, color $u_1u'_1$ with color 1; $v_1u_1$, $u'_1w_1$ (without loss of generality assume $v_1u_1, u'_1w_1 \in E(M)$), $f_1$ and $f'_1$ with colors 2 and 3 in an appropriate way. Then we get a 3-coloring, say $c'$, of $M$. Since each of $|H_1| - 1$, $|H_1| - 1$ and $|H_1| - 1$ is even, we can partition $f_4, \ldots, f_{2n+1}$ into $\frac{2n-3}{2}$ pairs in an appropriate way such that $c'(f_i) = c'(f_j)$ for each pair $(f_i, f_j)$. And let $S' = \{(e_i, e_j) \mid f_i$ and $f_j$ are in some pair$\}$. Suppose that $c'(f_1) = 2$ and $c'(f'_1) = 3$; then by identifying $f_1, f_2$ and $f'_1, f_3$ we get a 3-colorable cubic graph. By Lemma 2.3, there is a (2, 3)-path $P$ from $f_1$ to $f'_1$ in $M$. So we can interchange 2 and 3 on $P$, and get a new 3-coloring $c''$ of $M$; note that $c''(f_1) = 3$ and $c''(f'_1) = 2$. Now by identifying $f_1, f_3$ and $f'_1, f_2$ we also get a 3-colorable cubic graph. Thus let $S = S' \cup \{(e_1, e_2), (e_1, e_3)\}$, and then $S$ is the set required, which completes the proof. □

**Corollary 4.1.** *If $G$ is an uncolorable cubic graph, then $2s(G) - 1 \leqslant m(G) \leqslant 2s(G)$.*

**Proof.** Let $S$ be a set satisfying conditions (i), (ii), $\chi'(G(S)) = 3$ and $|S| = s(G)$. By Theorem 4.1, $|S| \geqslant \lceil \frac{m(G)}{2} \rceil$. If $|S| > \lceil \frac{m(G)}{2} \rceil$, by Theorem 4.2, one can find a set $S'$ such that $\chi'(G(S')) = 3$ and $|S'| = \lceil \frac{m(G)}{2} \rceil$. This is contrary to the definition of $s(G)$. So $|S| = \lceil \frac{m(G)}{2} \rceil$, and thus either $m(G) = 2|S|$ or $m(G) = 2|S| - 1$. □

**Remark 1.** Let $G$ be a cubic graph and $\chi'(G) = 4$. Then $\chi'(G(e_1, e_2)) = 3$ for some $e_1, e_2 \in E(G)$ if and only if $m(G) = 2$.

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